

# Elementary Excitations in a BEC with Isotropic Harmonic Trap: Bogoliubov Equations versus Hydrodynamic Formalism

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The elementary excitations for a BEC trapped by means of an isotropic harmonic oscillator are studied in the present work. The analysis of these perturbations is done in the context of the Bogoliubov equations and not resorting to the hydrodynamic version. The comparison between these two approaches will allow us to deduce a parameter explaining the role that the scattering length and the trap play in the way in which the frequency of the elementary excitations acquires information about the angular momentum of the corresponding solutions. It will be shown that outside the validity realm of the Thomas–Fermi approximation the frequencies of the perturbations cannot inherit the information of the angular momentum codified in the functions describing the elementary excitations.

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## I. INTRODUCTION

The analysis of the properties of elementary excitations in a Bose–Einstein Condensate (BEC) appears as an important aspect in the study of the properties of a BEC [1]. The reasons behind this interest range from the role played by collective modes and elementary excitations in the determination of thermodynamic properties [3] to the understanding of the phenomenon of superfluidity [2], among other possibilities. The analysis of these excitations offers, at least two options, namely, the hydrodynamic procedure [4], or the formalism known as Bogoliubov equations [5].

At this point we may wonder what kind of logical relation connects these two procedures, i.e., which one of them is more general, etc. Bogoliubov case can be used in the analysis of long wavelengths excitations (they correspond to sound waves) and short wavelengths (associated to free particle behavior) [6]. In other words, concerning the scale of length of these excitations the Bogoliubov method has no restrictions at all. In connection with the hydrodynamic analysis there are several approximations which are introduced, and clearly stated in the literature. For instance, the kinetic term appearing in the Gross–Pitaevski equation is separated into two contributions [3], namely, one related to the motion of particle and, a second one, corresponding to zero–point motion. The latter, known as quantum pressure term is neglected, as a consequence of the Thomas–Fermi approximation. and, therefore, the hydrodynamic analogue of the Gross–Pitaevski equation is obtained. This approach, relying on the Thomas–Fermi approximation, imposes a stringent condition upon the order of magnitude of the possible wavelengths of the excitations. Indeed, this model entails the fact that the kinetic energy is negligible compared against the other energies of the problem, a point

that implies that all phenomena related to scale lengths smaller than the so–called healing length [3] lie outside the validity realm of the hydrodynamic approach. In other words, the hydrodynamic model cannot provide us information about the case of wavelengths smaller than the healing length, i.e., only large wavelengths can be studied.

In addition, the deduction of the hydrodynamic equations assumes an additional restriction, the one does not appear, usually, in the literature, and that we, here, for the sake of completeness analyze. In order to explain this point in the clearest way we will resort to a usual deduction of this analogy, see pages 167 to 169 of [3]. We start with the time–dependent Gross–Pitaevski equation

$$i\hbar \frac{\partial \psi(\vec{r}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{r}, t) + V_t(\vec{r}) \psi(\vec{r}, t) + U_0 |\psi(\vec{r}, t)|^2 \psi(\vec{r}, t). \quad (1)$$

In this last expression  $V_t(\vec{r})$  denotes the trap used for the confinement of the system, which for this particular case is given by

$$V_t(\vec{r}) = \frac{m\omega_0 r^2}{2}. \quad (2)$$

Afterwards the following transformation is introduced in the Gross–Pitaevski equation

$$\psi(\vec{r}, t) = \sqrt{n(\vec{r}, t)} \exp(i\phi(\vec{r}, t)). \quad (3)$$

In this last expression  $n(\vec{r}, t)$  and  $\phi(\vec{r}, t)$  are real–valued functions. Then, we obtain two equations, one for the real part of Gross–Pitaevski and the second one for the imaginary term.

$$\frac{\partial n^2}{\partial t} = -\frac{\hbar}{m} \nabla \cdot (n^2 \nabla \phi), \quad (4)$$

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$$-\hbar \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m\sqrt{n}} \nabla^2 \sqrt{n} + \frac{1}{2}mv^2 + V_t + nU_0, \quad (5)$$

$$\vec{v} = \frac{\hbar}{m} \nabla \phi, \quad v = |\vec{v}|. \quad (6)$$

We now take the gradient of the last expression and obtain the motion equation for the velocity, namely,

$$m \frac{\partial \vec{v}}{\partial t} = -\nabla \left( V_t + nU_0 + \frac{1}{2}mv^2 - \frac{\hbar^2}{2m\sqrt{n}} \nabla^2 (\sqrt{n}) \right). \quad (7)$$

At this point the consequences of the Thomas–Fermi approximation are introduced, in the sense that the kinetic energy is neglected and therefore the quantum pressure term  $(\frac{\hbar^2}{2m\sqrt{n}} \nabla^2 \sqrt{n})$  is discarded. We now cast this last equation in a different way, the idea is to introduce the concept of pressure in our formalism. In the case of a uniform Bose gas the density  $n$  is a constant and, therefore, the energy of the system is  $E = (N-1)NU_0/(2V)$ , here  $N \gg \gg 1$  is the number of particles, whereas  $V$  is the corresponding volume. Since  $p = -\frac{\partial E}{\partial V}$  then for the homogeneous case  $p = \frac{n^2 U_0}{2}$ , a fact that implies a motion equation for the velocity with the following structure

$$m \frac{\partial \vec{v}}{\partial t} = -\nabla \left( V_t + \frac{1}{2}mv^2 \right) - 2\nabla \left( \frac{p}{n} \right). \quad (8)$$

Since  $n$  is constant then  $\nabla \left( \frac{p}{n} \right) = \frac{1}{n} \nabla p$ . But this expression is generalized to those cases in which the density ( $n$ ) is not a constant, a fact that entails an approximation. In addition, the usual analogy (see, equation (7.24) in [3]) assumes that the energy of the condensate is the same as that emerging in a homogeneous gas. Indeed, if we, within the Thomas–Fermi approximation, consider the energy of the condensate, then we must include not only the mean field energy ( $E = N^2 U_0/(2V)$ ) but also the energy stemming from the trapping potential. We may estimate this last energy as follows: the energy of one of the particles of the gas, due to the interaction with the isotropic harmonic oscillator, is  $m\omega_0^2 R^2/2$ , here  $R$  denotes the size of the condensate. Clearly,  $V = R^3$ , therefore, for  $N$  particles the whole energy reads  $E = N^2 U_0/(2V) + Nm\omega_0^2 V^{2/3}/2$ . Then the pressure becomes  $p = n^2 U_0/2 - nm\omega^2 V^{2/3}/3$ . Therefore, the correct expression, containing the pressure, is given by

$$m \frac{\partial \vec{v}}{\partial t} = -\nabla \left( V_t + \frac{1}{2}mv^2 \right) - \nabla \left( 2\frac{p}{n} + \frac{2m\omega^2 V^{2/3}}{3} \right). \quad (9)$$

This last expression becomes

$$m \frac{\partial \vec{v}}{\partial t} = -\nabla \left( V_t + \frac{1}{2}mv^2 \right) - \frac{2}{n} \nabla p + \frac{2p}{n^2} \nabla n. \quad (10)$$

The approximation introduced in the case of non-homogeneous gases is then

$$\frac{2p}{n^2} \nabla n = \frac{1}{n} \nabla p. \quad (11)$$

This last condition defines a functional dependence for  $p$  as a function of  $n$ .

$$p \sim n^2. \quad (12)$$

It is a rough approximation, remember that previously it was found that  $p = n^2 U_0/2 - nm\omega^2 V^{2/3}/3$ . In terms of characteristic lengths we may state that this approximation implies that the distance over which the pressure has a meaningful change is twice the corresponding distance for the density. This last statement can be understood noting that the approximation implies  $n^2 U_0/2 \gg \gg nm\omega^2 V^{2/3}/3$ . We now recall that under the presence of mean field interaction [4] the size of the condensate is given by  $R = (Na/\tilde{l})^{1/5} \tilde{l}$ , where  $a$  is the scattering length and  $\tilde{l} = \sqrt{\hbar/(m\omega_0)}$ . Joining these last two conditions we find that it is a good approximation at those points where the density  $n \gg (Na/\tilde{l})^{2/5}/(a\tilde{l}^2)$ . Therefore, at those points where the density becomes smaller than  $(Na/\tilde{l})^{2/5}/(a\tilde{l}^2)$  the approximation is not a good assumption. We now estimate this fact resorting to the Thomas–Fermi condition in which  $n = m\omega_0(R^2 - r^2)/U_0$  [3]. Under these conditions we find that the validity of the approximation implies that  $\tilde{l}^2(1 - 8\pi) > r^2$ , which is not possible. The conclusion is that it is not a good assumption for the case of an isotropic harmonic oscillator. These arguments show that the analogy between the condensate at  $T = 0$  and hydrodynamics contains not only the Thomas–Fermi approximation but additional assumptions.

A careful look at the Bogoliubov equations [6] allows us to state that there are no assumptions as those implicit in the hydrodynamic formalism. In other words, the solutions obtained from the former model will provide a better description than those stemming from the latter.

These arguments provide a motivation for the quest of solutions resorting to Bogoliubov equations. This is the issue addressed here. The corresponding solutions will be found and it will be shown that for those cases in which the angular momentum does not vanish there are an infinite number of frequencies for the elementary excitations, just as in the hydrodynamic model [3]. In addition, in contrast to the known situation [1], it will be shown that the mathematical structure of the our solutions is not a polynomial, but an infinite series. Of course, the convergence neighborhood is analyzed as a function of the properties of the condensate.

One of our main results will be related to the way in which the single-particle properties impinge upon the features of the elementary excitations. Indeed, each one of the particles conforming the gas (the depletion term is here neglected) lies, due to the lowness of the temperature, in the ground state of its one-particle Hamiltonian, i.e., a state with vanishing angular momentum. When the interaction due to the mean field contribution (this term measures the strength of the interaction that a particle in the BEC experiences as a consequence of the presence of the remaining particles in the gas) is not strong enough then the characteristics of the frequencies of the elementary excitations can acquire only those features belonging to the single-particle realm. The threshold defining *not enough* is also analyzed.

## II. BOGOLIUBOV EQUATIONS

The starting point is the time-dependent Gross-Pitaevski equation

$$i\hbar \frac{\partial \psi(\vec{r}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{r}, t) + m\omega_0^2 r^2 \psi(\vec{r}, t)/2 + U_0 |\psi(\vec{r}, t)|^2 \psi(\vec{r}, t). \quad (13)$$

Now a change in the order parameter ( $\psi(\vec{r}, t)$ ) is introduced in this last equation, namely,

$$\delta\psi(\vec{r}, t) = \exp(-it\mu/\hbar)[u(\vec{r}) \exp(-i\omega t) - v^*(\vec{r}) \exp(+i\omega t)]. \quad (14)$$

Then we obtain two equations, known as Bogoliubov equations [5, 6].

$$-\frac{\hbar^2}{2m} \nabla^2 + 2nU_0 - \hbar\omega - \mu + m\omega_0^2 r^2/2] u(\vec{r}) = [\mu - m\omega_0^2 r^2/2] v(\vec{r}), \quad (15)$$

$$-\frac{\hbar^2}{2m} \nabla^2 + 2nU_0 + \hbar\omega - \mu + m\omega_0^2 r^2/2] v(\vec{r}) = [\mu - m\omega_0^2 r^2/2] u(\vec{r}). \quad (16)$$

They are two coupled differential equations, and here we proceed to uncouple them. Defining

$$\begin{aligned} \hat{L} &= -\frac{\hbar^2}{2m} \nabla^2, \\ \hat{H} &= \mu - m\omega_0^2 r^2, \\ f(\vec{r}) &= v(\vec{r}) + u(\vec{r}), \\ F(\vec{r}) &= u(\vec{r}) - v(\vec{r}). \end{aligned} \quad (17)$$

We may cast the Bogoliubov equations as

$$\hat{L}f(\vec{r}) = \hbar\omega F(\vec{r}), \quad (18)$$

$$\hat{L}F(\vec{r}) = \hbar\omega f(\vec{r}) - 2\hat{H}F(\vec{r}). \quad (19)$$

Finally, from these last two equations we find for our function  $F(\vec{r})$  the following condition

$$\hat{L}[\hat{L}F(\vec{r}) + 2\hat{H}F(\vec{r})] = (\hbar\omega)^2 F(\vec{r}). \quad (20)$$

The structure of our two equations allow us to seek our solutions with the form

$$F(\vec{r})_{(l)}^{(\tilde{m})} = R(r)_{(l)} Y(\theta, \phi)_{(l)}^{(\tilde{m})}. \quad (21)$$

Here  $Y(\theta, \phi)_{(l)}^{(\tilde{m})}$  denote the spherical harmonics and, in consequence, the parameters  $l$  and  $\tilde{m}$  are the angular momentum of the excitation and its  $z$ -component, respectively. This last assumption allows us to end up with a fourth-order differential equation.

$$\begin{aligned} & \frac{\hbar^4}{4m^2} \left[ \frac{d^4 R_{(l)}}{dr^4} + \frac{4}{r} \frac{d^3 R_{(l)}}{dr^3} \right] \\ & + \frac{\hbar^2}{2m} \left[ m\omega_0^2 r^2 - 2\mu - \frac{l(l+1)\hbar^2}{2mr^2} \right] \frac{d^2 R_{(l)}}{dr^2} \\ & + \frac{\hbar^2}{2m} \left[ 6m\omega_0^2 r - \frac{4\mu}{r} + \frac{l(l+1)\hbar^2}{mr^3} \right] \frac{dR_{(l)}}{dr} \\ & + \frac{\hbar^2}{2m} \left[ 6m\omega_0^2 - \frac{l(l+1)\hbar^2}{2mr^4} \right] R_{(l)} = (\hbar\omega)^2 R_{(l)}. \end{aligned} \quad (22)$$

At this point already a remarkable difference emerges when we compare this last equation against the fundamental expression within the hydrodynamic formalism. Indeed, the motion equation for the elementary excitations (as a function of the perturbations in the density, i.e.,  $\delta n$ ) is a second-order differential equation [3]

$$m \frac{\partial^2 \delta n}{\partial t^2} = \frac{1}{m} [\nabla(m\omega_0^2 r^2/2) \cdot \nabla \delta n - (\mu - m\omega_0^2 r^2/2) \nabla^2 \delta n]. \quad (23)$$

## III. SOLUTIONS TO BOGOLIUBOV EQUATIONS

Our main result is contained in (22) and now we take the simplest situation, namely,  $l = 0$ . It is readily seen that a solution exist if

$$R_{(l=0)} = \text{constant}, \quad \omega = \sqrt{3}\omega_0. \quad (24)$$

The corresponding structure for the elementary excitations is

$$\delta\psi(\vec{r}, t) = \exp(-it\mu/\hbar) \frac{R_{(l=0)}}{2} \left[ (1 + g(r)) \exp(-i\sqrt{3}t\omega_0) - (-1 + g(r)) \exp(i\sqrt{3}t\omega_0) \right], \quad (25)$$

$$g(r) = \frac{2\mu}{\sqrt{3}\hbar\omega_0} - \frac{1}{\sqrt{3}} \left( \frac{r}{\tilde{l}} \right)^2, \quad (26)$$

$$\tilde{l} = \sqrt{\frac{\hbar}{m\omega_0}}. \quad (27)$$

We now proceed to consider the solution for  $l > 0$ . This will be done in a different spirit to the analysis of the hydrodynamic situation. In the latter the allowed frequencies are deduced imposing a stringent condition upon the general solution of the corresponding differential equation. In the case of the hydrodynamic model the corresponding equation, for  $l \neq 0$ , is the hypergeometric equation [3], and it is reduced to a polynomial [1] imposing some conditions upon the parameters of the solution, see page 181 [3]. Of course, the reduction of the hypergeometric function to a polynomial guarantees the convergence of the solution at all points inside the condensate. This procedure renders the frequencies for the excitations as a function of the angular momentum (values of  $l$ ) and the number of radial nodes of the emerging polynomials, see expression (7.71) page 181 [3]. It must be stressed that the reduction of the hypergeometric function to a polynomial is a sufficiency condition for the convergence of the solution, though not a necessity condition.

Here we consider the solution as an infinite series and look for the conditions that ensure the convergence of the solution at all points inside the condensate.

Our solution has the form (here the angular momentum does not vanish, i.e.,  $l = 1, 2, \dots$ )

$$R_{(l)} = \sum_s b_s r^s. \quad (28)$$

Introducing this function into (22) we obtain the following conditions

$$l(l+1)b_0 = 0, \quad (29)$$

$$l(l+1)b_2 = 0, \quad (30)$$

$$\frac{\hbar^2}{m} (6 - l(l+1))b_3 = 2\mu b_1, \quad (31)$$

$$\begin{aligned} & \frac{\hbar^4}{4m^2} ([s+5][s+4][s+3][s+2] - \\ & l(l+1)[s+3]^2) b_{(s+4)} - \frac{\hbar^2\mu}{m} [s+3][s+2] b_{(s+2)} \\ & + \hbar^2 (\omega_0^2 [s+3][s+2] - \omega) b_{(s)}, \quad s = 0, 1, 2, \dots, \end{aligned} \quad (32)$$

Concerning (29)–(31) we must add that they are imposed to discard all kind of singularities. The last one is the recursion relation for the coefficients of our solution. Notice that these last conditions imply that all the coefficients of the type  $b_{(2s)}$  must vanish. In other words, in contrast with the situation of the hydrodynamic formalism, where the solutions are even functions of the radial coordinate ([1]), here only odd powers of  $r$  emerge.

In order to illustrate the behavior of our solutions we analyze the case  $l = 2$ . Under this condition (29)–(32) imply

$$b_1 = 0, \quad (33)$$

$$b_5 = \frac{48m\mu}{264\hbar^2} b_3. \quad (34)$$

From the recursion condition (32) we find the allowed frequencies. Indeed, notice that this expression allows us consider an  $\omega$  for each  $s = 3, 5, 7, \dots$  as follows

Consider (32) for  $s = 3$

$$\frac{366\hbar^4}{4m^2} b_{(7)} - \frac{30\hbar^2\mu}{m} b_{(5)} + \hbar^2 (30\omega_0^2 - \omega) b_{(3)} = 0. \quad (35)$$

Here we impose the condition

$$30\omega_0^2 - \omega. \quad (36)$$

Then (35) becomes

$$b_{(7)} = \frac{30\mu}{366\hbar^2 m} b_{(5)}. \quad (37)$$

In other words, we obtain the frequency and the solution, with just one free parameter,  $b_{(3)}$

$$\begin{aligned} R_{(l=2)} &= b_{(3)} r^3 \left[ 1 + \frac{48}{364} \frac{m\mu}{\hbar^2} r^2 \right. \\ & \quad \left. + \frac{48}{364} \frac{30}{264} \left( \frac{m\mu}{\hbar^2} \right)^2 r^4 + \dots \right]. \end{aligned} \quad (38)$$

Let us now address the issue concerning the convergence of this series. Our expressions imply that

$$b_{(2s+1)} = (10^{-1})^s \left( \frac{m\mu}{\hbar^2} \right)^{s-3} \left[ \left( \frac{m\mu}{\hbar^2} \right)^2 - \frac{1}{10\tilde{l}^4} \right] b_{(3)}. \quad (39)$$

A necessary but not sufficient condition [7] for the convergence of this series reads

$$b_{(2s+3)} r^{(2s+3)} / (b_{(2s+1)} r^{(2s+1)}) \rightarrow 0, \quad \text{if } s \rightarrow \infty. \quad (40)$$

For our case this condition entails

$$r^2 < \frac{10\hbar^2}{m\mu}. \quad (41)$$

In order to evaluate this condition in terms of the parameters of the condensate let us consider the value of the chemical potential according to the Thomas–Fermi approximation ( $\mu = m\omega_0^2 R^2/2$  with the relation  $R = (Na/\tilde{l})^{1/5} \tilde{l}$  [3]). Under these conditions (41) becomes

$$r < \sqrt{20} \left( \frac{\tilde{l}}{Na} \right)^{1/5} \tilde{l}. \quad (42)$$

A convergence radius, at least equal to  $R$ , requires

$$R \leq r. \quad (43)$$

Therefore, our solution is valid in the whole condensate if

$$(Na/\tilde{l})^{2/5} < \sqrt{20}. \quad (44)$$

This last expression provides the necessity requirements needed to have a convergent series at all points within the condensate, as a function of its parameters.

The sufficiency condition that guarantees the convergence of our solution requires [7] that  $\forall \epsilon > 0$ , there exists an integer  $M$  such that

$$\sum_{s=M}^{\infty} b_{(2s+1)} r^{(2s+1)} < \epsilon. \quad (45)$$

We may obtain an upper bound for this expression, since (41) is fulfilled.

$$\sum_{s=M}^{\infty} b_{(2s+1)} r^{(2s+1)} \leq \frac{\left( \frac{m\mu r^2}{10\hbar^2} \right)^{M-1}}{1 - \frac{m\mu r^2}{10\hbar^2}} b_{(3)}. \quad (46)$$

It readily seen that we may find an integer  $M$  such that

$$\left( \frac{m\mu r^2}{10\hbar^2} \right)^{M-1} < \frac{\epsilon}{b_{(3)}} \left( 1 - \frac{m\mu r^2}{10\hbar^2} \right). \quad (47)$$

Inserting (47) into (46) we conclude that the series does converge. In other words, for this particular situation, the necessity condition turns out to be a sufficiency condition.

Additional frequencies and solutions can be found as follows. The frequency has been obtained, in the present case, from the recursion expression (32) imposing the condition  $\omega_0^2(s+3)(s+2) - \omega^2 = 0$ , for  $s = 3$ . More frequencies emerge if we demand the fulfillment of this condition for  $s = 4, 5, 6, \dots$ . The convergence of the corresponding series is guaranteed by the structure of the recursion expression.

#### IV. BOGOLIUBOV EQUATIONS VERSUS HYDRODYNAMIC FORMALISM

Notice that our solution for  $l = 0$  (see expressions (25) and (26)) describes a mode which is more localized near the surface of the cloud, i.e., it corresponds to surface waves. The mathematical structure of our solutions, series in the radial coordinate, tell us that this last characteristic of the case  $l = 0$  is shared by all the solutions, namely, they are related to surface waves. We may then state that for particle-like modes the elementary excitations of a BEC trapped by an isotropic harmonic oscillator are surface waves. Of course, due to the spherical symmetry of the solutions we are in the presence of degenerated waves. In addition, the higher the frequency becomes, the more localized near the surface that these waves are.

Our formalism for the case of vanishing angular momentum does not coincide with the results of the hydrodynamic procedure, in which the frequency is given by  $\omega(s, l) = \omega_0(2s^2 + 2sl + 3s + l)^{1/2}$ , here  $s$  denotes the number of radial nodes of the solution [1], i.e., it has an infinite number of frequencies related to the case  $l = 0$ . If  $l = 0$ , then  $\omega(s, 0) = \omega_0(2s^2 + 3s)^{1/2}$ .

Our frequency, just one possible value, is given by  $\omega = \sqrt{3}\omega_0$ , a result which does not coincide with any of the possible cases of the hydrodynamic formalism. As a matter of fact, our frequency is smaller than all the possible values of the other model, whose smallest value reads ( $s = 0$  is discarded in this discussion)  $\omega(s = 1, l = 0) = \sqrt{5}\omega_0$ . In other words, for vanishing angular momentum our assertion is that collective modes have always larger frequencies than those of other possible modes.

We may explain this fact as a consequence of the role that the kinetic energy term plays in the definition of the dynamics of these elementary excitations. Indeed, resorting to a sum rule approach [8] we find that the introduction in the hydrodynamic procedure of the effects of the kinetic energy term produces a reduction

of the frequency the one depends upon the ratio of the energy related to the harmonic oscillator ( $E_{(ho)}$ ) and the kinetic energy ( $E_{(kin)}$ ) [1], which for the case of  $s = 1$  and vanishing angular momentum takes the form  $\omega^2 = \omega_0^2(5 - E_{(kin)}/E_{(ho)})$ . This last fact explains our result, indeed, since our method does not discard the effects of the kinetic energy we should expect a lower frequency than that related to the smallest frequency of the hydrodynamic formalism. In this case it corresponds to  $E_{(kin)}/E_{(ho)} = 2$ , a fact that confirms that we are outside the validity region of the Thomas–Fermi approximation.

Another interesting point of our result stems from the fact that the frequencies are  $l$ -independent, see (32), though the solutions do depend upon the value of the angular momentum. This dependence can be seen at the recursion relation where  $l$  appears explicitly. In the solutions of the hydrodynamic model both, frequencies and solutions, show an explicit dependence upon the corresponding value of the angular momentum.

We may interpret this results as follows. The smallness of the parameter  $(Na/\tilde{l})^{1/5}$ , required for the convergence of our solutions ( $(Na/\tilde{l})^{2/5} < \sqrt{20}$ ), entails that the strength of the repulsive interaction (codified here in the factor  $Na$ ) is not strong enough in order to allow the frequency of the elementary excitations to be determined as a function of the angular momentum of the proposed solution (21). In other words, the presence of the trap and of the scattering length imply that the ensuing solutions depend upon these aforementioned parameters, though the frequency does depend upon the angular momentum (as happens in the hydrodynamic approach) only if the parameter  $(Na/\tilde{l})^{1/5}$  is beyond a certain threshold, which for our case can be estimated to be  $\sqrt{20} \leq (Na/\tilde{l})^{2/5}$ . In other words, it is the repulsive interaction the one responsible for the appearance in

the frequency of the elementary excitations of the angular momentum of the corresponding solution. We may then generalize this last conclusion as a conjecture: In a trapped BEC the frequency of the elementary excitations acquire information of the angular momentum of the solution only by means of the parameter  $(Na/\tilde{l})^{1/5}$ , and this happens only if this parameter is beyond a certain threshold, in the present case it is, approximately,  $\sqrt{20} \leq (Na/\tilde{l})^{2/5}$ . This last condition can be understood from a different perspective. Indeed, each one of the particles conforming the gas (the depletion term is here neglected) lies, due to the lowness of the temperature, in the ground state of its one-particle Hamiltonian, i.e., a state with vanishing angular momentum. Since the interaction due to the mean field contribution (this term measures the strength of the interaction that a particle in the BEC experiences as a consequence of the presence of the remaining particles in the gas) is not strong enough then the characteristics of the frequencies of the elementary excitations can depend only from those features belonging to the single-particle realm. A further argument in this direction can be seen in the fact that in the hydrodynamic approach the case  $l = 1$  corresponds to a translation of the cloud with no change in its internal structure [3]. This kind of motion involves a bulk movement of the particles of the BEC. Notice that in our case this kind of effect is absent, and the reason lies in the fact that the repulsive interaction is not strong enough and, therefore, no bulk properties can emerge.

If we consider any value of the allowed frequency associated to the case here explicitly shown ( $l = 2$ ) we find from the recursion expression (32) that it emerges for any other non-vanishing value of  $l$ . Our last comment also explains this degeneracy of the frequencies, namely, they are obtained only from the term whose coefficient is  $\hbar^2$  in (32), i.e., it does not contain  $l$ .

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